A New Simplicial Cover Technique in Constrained Global Optimization

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Abstract. A simplicial branch and bound-outer approximation technique for solving nonseparable, nonlinearly constrained concave minimization problems is proposed which uses a new simplicial cover rather than classical simplicial partitions. Some geometric properties and convergence results are demonstrated. A report on numerical aspects and experiments is given which shows that the most promising variant of the cover technique can be expected to be more efficient than comparable previous simplicial procedures.

Key words. Constrained global optimization, concave minimization, simplicial covers, branch and bound, outer approximation

1. Introduction and Brief Survey of Existing Methods

The purpose of this article is to present a new algorithm for solving the concave minimization problem

(P) minimize $f(x)$, subject to $x \in D$,

where f is a concave, not necessarily separable real-valued function on a suitable open convex set $A \subset \mathbb{R}^n$ containing D, and $D := \{x \in \mathbb{R}^n : g_i(x) \leq 0 \ (i = 1, 2, \ldots, l)\}$. We will assume, for each $i \in \{1, 2, \ldots, l\}$ that g_i is a real-valued convex function on A, that D is a nonempty, compact set, and that there exists a point $p \in \mathbb{R}^n$ such that $g_i(p) < 0$ $(i = 1, 2, \ldots, l)$ (Slater condition). By introducing the convex function $g(x) := \max\{g_i(x) : i = 1, \ldots, l\}$, we rewrite problem (P) in the form

(P) minimize $f(x)$, subject to $g(x) \leq 0$.

Problem (P) may have many locally-optimal solutions which are not globally optimal. While it is well known that a globally optimal solution for problem (P) exists which is an extreme point of D, D may have an infinite number of extreme points. Let $m := \min f(D)$ denote the optimal objective function value for problem (P).

Numerous applications from many different fields lead to concave minimization problems. In addition, several other difficult problems of interest in optimization can be transformed into equivalent concave minimization problems; examples

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include bilinear programming problems, linear and concave complementarity problems, certain max-min problems and integer programming problems. Moreover, most successful concave minimization techniques can be extended to the considerably more general d.c. programming problem, where now all of the functions involved in (P) can be represented as the difference of two convex functions.

A comprehensive treatment of concave minimization, d.c. programming, and other global optimization problems and techniques is given in the recent monograph of Horst and Tuy (1990). Therefore, for the development of concave minimization until 1989, we refer the reader to this monograph and its numerous references. We restrict ourselves here to a brief outline of those methods which are directly relevant to our new approach and to numerical and methodological results which are not (fully) reported in Horst and Tuy (1990). Note that we are interested here in algorithms for solving (P) in the case of nonlinear g_i (i.e., nonpolyhedral feasible sets D), and we do not assume separability of the functions involved.

The first convergent algorithm for solving the nonlinearly constrained concave minimization problem (P) was the branch and bound procedure of Horst (1976): an initial n-simplex $M_0 \supset D$ is refined by a partitioning procedure which, by replacing one vertex of one of the longest edges of a simplex by the midpoint of this edge, successively subdivides certain most promising subsimplices of M_0 into two smaller simplices of equal volume. This refining procedure is frequently called a bisection. On each simplex M generated by the procedure one computes the uniformly best convex underestimating function φ_M of f on M (the convex envelope of f on M). By minimizing the convex envelopes over $M \cap D$, for each M of a current partition of a certain subset of M_0 still of interest, one obtains lower bounds for the optimal value m of (P) which eventually converge to m (Horst, 1976 and 1980), see also Horst and Tuy (1990).

The convex envelope φ_M of f on M is the uniquely defined affine function that coincides with f at the vertices of M immediately available when barycentric coordinates relative to M are used: Let $M = \operatorname{conv}\{v^0, \ldots, v^n\}$ be given as convex hull of its $n+1$ vertices $v^0, \ldots v^n$. Then every $x \in M$ is uniquely representable in the form

$$x = \sum_{i=0}^{n} \lambda_i v^i,\ \sum_{i=0}^{n} \lambda_i = 1,\ \lambda_i \geq 0\ (i = 0, \ldots, n) \text{ and } \varphi_M(x) = \sum_{i=0}^{n} \lambda_i f(v^i)\,.$$

In each branching step, however, two nonlinear convex programming problems of the type "minimize $\varphi_M(x)$ over $M \cap D$" have to be solved. Numerical investigations reported in Thoai and de Vries (1988), Horst, Thoai, and Benson (1991) and Horst (1991) show that it is not worthwhile to solve these convex problems very accurately in each step. A few dual LP-steps of a classical (slow) cutting-plane approach such as the supporting hyperplane method of Veinott (1967) often provides more "overall" efficiency than an accurate solution by means of a faster general purpose nonlinear programming algorithm such as SQP (sequential quadratic programming). But these numerical investigations also show that this simplicial approach is not very efficient for larger problem sizes whatever convex programming code is used. Apart from the computationally-expensive subproblems, a second drawback concerns the use of the simplex partitions which responds to the extreme point optimality of problem (P) only by an obvious deletion of all simplices $M \supset \text{int } D$. Note, however, that modifications and specializations of this approach are of interest and (or) are numerically quite efficient for a number of specially-structured problems, for example, in biconvex optimization (Al-Khayyal and Falk, 1983), in separable concave minimization (Falk and Soland, 1969; Pardalos and Rosen, 1987; Horst and Tuy, 1990, and references therein), and in certain interactive fixed charge problems (Benson and Erenguc, 1988).

A second well-known class of methods has been termed *outer approximation*: a decreasing sequence of polytopes $P_k \supset D$ is constructed and an optimal solution of (P) is successively approximated by a solution x^k of the relaxed problem "minimize $f(x)$ over P_k". If $x^k \in D$, then x^k is an optimal solution of (P). If $x^k \not\in D$, then one sets $P_{k+1} = P_k \cap \{x: l_k(x) \leq 0\}$, where l_k is a suitable real-valued affine function satisfying $l_k(x) \leq 0 \forall x \in D$ and $l_k(x^k) > 0$. This well-known scheme, which has been used in many fields of optimization, can be applied to problem (P) in various ways (see, for example, Hoffman, 1981; Thieu, Tam, and Ban, 1983; Horst, Thoai, and Tuy, 1987 and 1989). A general convergence theory, which includes various "constraint dropping strategies", generalization to nonlinear "cuts" $l_k(x)$ and many details is given in Horst and Tuy (1990). In our case of concave minimization, the approximations x^k are determined by enumeration of the vertices of P_k, and the computationally-crucial part concerns in the determination of all new vertices of the polytope P_{k+1} which is generated from the polytope P_k (whose vertex set is known) by a cutting hyperplane $\{x: l_k(x) = 0\}$. Various simple techniques are known for this enumeration problem (see Horst and Tuy, 1990; Chen, Hansen, and Jaumard, 1991 and references therein).

Outer approximation methods are often fast for small problems but not competitive with the best methods when the problem size increases (cf. Thoai and de Vries, 1988; Horst and Thoai, 1989). This behavior is due to the inevitable enumeration of the new vertices whose number is usually rapidly increasing with the dimension of D (e.g., Horst, Thoai, and de Vries, 1988).

The cone splitting method of Tuy, Thieu, and Thai (1985) combines some ideas from branch and bound and outer approximation: Assuming $0 \in \text{int } D$, an n simplex S satisfying $0 \in \text{int } S$ is constructed, and the collection of conical hulls $\{\text{cone } F_i: i = 1, \ldots, n\{$ of the $n+1$ facets F_i of S defines a conical partition of $\mathbb{R}^n$. Most promising cones are successively refined by bisecting the corresponding simplices F_i and forming the conical hulls of the corresponding subsimplices. For each cone C, at the intersection points of its n generating edges with the boundary ∂D of D, and at certain additional boundary points of D the supporting hyperplanes of D are constructed. The intersection of the cone C and all of the closed halfspaces corresponding to these hyperplanes is a polytope whose minimal vertex value of f defines the lower bounds in the conical branch and bound scheme. This approach, however, faces numerical vertex enumeration problems similar to the outer approximation methods and can handle only relatively small problem sizes (Thoai and de Vries, 1988).

Note that in simplicial and conical branch and bound methods one can apply various so-called radial subdivision rules rather than mere bisection (e.g., Horst and Tuy, 1990; Tuy, 1991). Often an acceleration results, but, in general, the handleable problem sizes cannot be increased significantly by modified radial subdivisions alone.

A considerable step forward in the direction of more efficient concave minimization algorithms has been achieved very recently with new methods that combine typical branch and bound elements like partitioning, deletion and bounding with suitably introduced outer approximation cuts in such a way that the computationally most expensive subroutines of the previous algorithms are avoided. In Horst, Thoai, and Benson (1991), a conical branch and bound-outer approximation technique is proposed which highly exploits the concavity of f in the bounding and deletion procedure. Horst and Benson (1991) propose a simplicial branch and bound-outer approximation technique, and Benson (1990) treats the separable concave minimization problem in a similar way via rectangular partitions. One of the major advantages of these algorithms is that the only significant nonlinear computation required at each iteration can be accomplished by any of a number of simple univariate search procedures. The only other major computations required involve only either linear programming subproblems and (in Benson and Horst, 1991) linear systems of $(n + 1)$ unknowns which have a unique solution. Numerical experiments reported in Horst, Thoai, and Benson (1991) indicate that clever implementations of the most promising of the above branch and bound-outer approximation ideas are considerably more efficient for larger problem sizes than the pure branch and bound and the pure outer approximation algorithms.

The motivation of the present article departs from the branch and bound-outer approximation algorithm of Benson and Horst (1991) which, in its branch and bound part, uses standard simplicial partitioning and convex underestimating functions in a similar way as in the above described original simplicial branch and bound approach of Horst (1976). Though the optimal solution of (P) is an extreme point of the feasible set D, many simplices are generated which have large parts in the interior of D, far away from any extreme point. As a consequence, much computation has to be carried out at points which are certainly not very good approximations of a global minimizer. Therefore, in Section 2 we introduce a new *simplicial covering* technique which allows immediate deletion of a large part of the interior of D, and provides considerably-improved initial bounds for the optimal value m. A related refining strategy and a number of new results on its geometry and convergence properties are presented, which include sufficient conditions for the repeated application of this cover to be exhaustive.

Section 3 introduces a slight generalization of the so-called γ-extension, which until now has been successfully used only in conical algorithms (cf. Horst and Tuy, 1990; Horst, Thoai, and Benson, 1991 and references therein). This generalization will allow one to incorporate the γ-extension concept in the simplicial cover procedure. The new algorithm is presented which generates sequences of upper and of lower bounds for the optimal value m of (P). It is proved that, when the algorithm is not finite, every accumulation point of both the sequence of lower and the sequence of upper bounds equals m. As a side effect, it results that in the Benson–Horst (1991) approach, where only convergence of the lower bounds is proved, we also can introduce upper bounds converging to m.

Finally, Section 4 illustrates the new algorithm using the example of Benson–Horst (1991), and reports on implementation and numerical comparison with the Benson–Horst method.

2. A Class of Simplicial Covers

In this section a class of *simplicial covers* is introduced which, in contrast to the classical simplicial *partition* (cf. Horst and Tuy, 1990), allows overlap of interior part of the simplices involved. A number of relevant geometric and convergence properties is discussed, which includes sufficient conditions for the infinitely repeated application of the cover to be exhaustive. Some of the very involved proofs, however, have to be omitted here. These proof and additional results will be published in a separate article (Horst, Thoai, and de Vries, 1991).

DEFINITION 2.1. Let M, M_i $(i \in I \subset \mathbb{N})$ be n dimensional simplices (n-simplices) in $\mathbb{R}^n$ satisfying $M_i \cap M \neq \emptyset$ $(i \in I)$. The collection $\mathcal{M} := \{M_i : i \in I\}$ is said to be a *simplicial cover* of M if

$$M \subset \bigcup_{i \in I} M_i ,$$

$$M_j \not\subset \bigcup_{\substack{i \in I \\ i \neq j}} M_i (j \in I) .$$

The following lemmas and theorems introduce a class of simplicial covers. Let conv A and aff A denote the convex hull and the affine hull, respectively, of a set $A \subset \mathbb{R}^n$.

LEMMA 2.1. *Let $M = \operatorname{conv}\{v^0, \ldots, v^n\}$ be an n-simplex in $\mathbb{R}^n$ with vertices $v^0, \ldots, v^n$, and let $x^0 \in \operatorname{int} M$. Then, for arbitrary $\mu_i > 1$ $(i = 0, \ldots, n)$, the points*

$$s^i = v^i + \mu_i(x^0 - v^i) \quad (i = 0, \ldots, n) \tag{1}$$

are affinely independent, i.e, $\bar{M} := \mathrm{conv}\{s^0, \ldots, s^n\}$ *is an n-simplex. Moreover, we have* $x^0 \in \mathrm{int}\, \bar{M}$.

Proof. Without loss of generality assume $x^0 = 0$. Then, by (1) a linear mapping

$$T: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad s = Tv$$

is defined. The mapping T is regular since the matrix T is a diagonal matrix with diagonal elements $1 - \mu_i < 0$ $(i = 0, , \ldots, n)$. It follows that the images s^i of the affinely independent points v^i $(i = 0, \ldots, n)$ are affinely independent, and hence $\bar{M}$ is an n-simplex. The restriction of T on M is an isomorphism $M \to \bar{M}$ (with respect to the Euclidean topology on M, resp. $\bar{M}$) which maps the interior point $x^0 = 0$ onto the interior point $0 = T(0)$ of $\bar{M}$.

LEMMA 2.2. *Let* v^i *and* s^i $(i = 0, \ldots, n)$ *be defined as in Lemma 2.1. Then, for all* $i \in \{0, \ldots, n\}$, *the hyperplane*

$$H_i := \mathrm{aff}\{s^0, \ldots, s^{i-1}, s^{i+1}, \ldots, s^n\} \tag{2}$$

strictly separates v^i *from*

$$\mathrm{conv}\{x^0, v^j (j = 0, \ldots, n;\ j \neq i)\}$$

whenever $H_i \cap M \neq \emptyset$ *and* $v^i \not\in H_i$.

A proof of Lemma 2.2 is given in Horst, Thoai, and de Vries (1991).

THEOREM 2.1. *Let* $M = \{x \in \mathbb{R}^n : g_i(x) \leq 0 \ \ (i = 0, \ldots, n)\}$ *be the inequality-representation of an n-simplex M with facets* $F_i := M \cap \{x : g_i(x) = 0\}$ $(i = 0, \ldots, n)$ *and vertices* $v^0, \ldots, v^n$. *Let* $x^0 \in \mathrm{int}\, M$, *and denote by* v^i *the vertex of M opposite to* F_i *(i.e.,* $v^i \not\in F_i$)$(i = 0, \ldots, n)$. *Moreover, let*

$$s^i = v^i + \mu_i(x^0 - v^i), \quad \mu_i > 1 \quad (i = 0, \ldots, n).$$

Finally, let

$$H_i = \mathrm{aff}\{s^j : j = 0, \ldots, n; j \neq i\} = \{x \in \mathbb{R}^n : h_i(x) = 0\} \quad (i = 0, \ldots, n)$$

with the suitable affine functions h_i *defined such that* $h_i(x^0) > 0$ $(i = 0, \ldots, n)$. *Then*

$$\mathcal{M} := \{M_i : h_i(v^i) < 0 \quad (i \in 0, \ldots, n)\} \tag{3}$$

where

$$M_i = \{x \in \mathbb{R}^n : g_j(x) \leq 0 \ (j = 0, \ldots, n;\ j \neq i), \quad h_i(x) \leq 0\}$$
$$(i = 0, \ldots, n) \tag{4}$$

and

$$M_{n+1} = \bar{M} = \{x \in \mathbb{R}^n : h_j(x) \geq 0 \quad (j = 0, \ldots, n)\}, \tag{5}$$

defines a simplicial cover of M.

Proof. The set M_{n+1} is the simplex $\bar{M}$ in Lemma 2.1. The sets M_i satisfying $h_i(v^i) < 0$, i.e., $H_i \cap M \neq \emptyset$, $v^i \not\in H_i$, are n-simplices because of Lemma 2.2.

Let I denote the set of indices $i \in \{0, \ldots, n\}$ with $M_i \in \mathcal{M}$. In order to demonstrate $M \subset \cap^{i \in I} M_i$, let $x \in M$. Then we either have $h_i(x) \geq 0$ ($i = 0, \ldots, n$), and hence $x \in M_{n+1}$, or else there exists $i \in \{0, \ldots, n\}$ such that $h_i(x) < 0$, and hence $x \in M_i$.

Because of (4) we have $v^i \in M_i$ ($i = 0, \ldots, n$), but from Lemma 2.2 we know that $h_j(v^i) > 0$ ($j \neq i$) (since $h_j(x^0) > 0$), and hence $v^i \not\in M_j$ ($i \neq j$). Finally, $x^0 \in M_{n+1}$, but $h_i(x^0) > 0$ ($i = 0, \ldots, n$) implies $x^0 \not\in M_i$ ($i = 0, \ldots, n$), and we have shown that

$$M_j \not\subset \bigcup_{\substack{i \in I \\ i \neq j}} M_i \quad (j \in I). \qquad \square$$

REMARK. The condition $h_i(v^i) < 0$ in (3) is equivalent to $H_i \cap M \neq \emptyset$, $v^i \not\in H_i$, and it is easy to see that, when $h_i(v^i) \geq 0$ for $i \in I$, the number of simplices in $\mathcal{M}$ is $n + 2 - |I|$. Note that the simplex M_{n+1} in Theorem 2.1 coincides with the simplex $\bar{M}$ in Lemma 2.1. In the algorithm for solving problem (P) which we present in the next section, the points s^i will always be chosen in such a way that the simplex $M_{n+1} = \bar{M}$ can be deleted from further consideration, because we will know that the minimum value m of (P) cannot be attained here.

In order to demonstrate the convergence of the algorithm which follows in Section 3, it is necessary to investigate the behaviour of decreasing sequences $\{M_q\}$ of simplices, where M_{q+1} is an element of the cover of M_q which we introduced in Theorem 2.1, and which never has an element of the form $\bar{M}$. Referring to the general convergence theory of branch and bound methods developed in Horst (1986), Horst and Tuy (1987), Tuy and Horst (1988) (see Horst and Tuy, 1990 for a detailed exposition), we are particularly interested in conditions which ensure that the M_q converge to a singleton. Note that the above convergence theory uses partitions rather than covers, where a simplicial partition $\mathcal{M} = \{M_i\colon M_i$ is an n-Simplex, $i \in I\}$ of an n-simplex M is defined by the conditions $M = \cup_{i \in I} M_i$, $\operatorname{int} M_i \cap \operatorname{int} M_j = \emptyset$ ($i \neq j$). (Clearly, a partition is a cover). However, it is easy to see that this theory still holds when we replace partitions by covers.

Let $\delta(M)$ denote the diameter of a simplex M, which is the length of a longest edge of M.

DEFINITION 2.2. A successive simplicial cover is called *exhaustive* if $\delta(M_q) \xrightarrow[q \to \infty]{} 0$ *for all decreasing sequences* $\{M_q\}$ of simplices, where M_{q+1} is an element of the cover of M_q.

The quite involved proof of the following result pertaining to exhaustiveness of our cover is given in Horst, Thoai, and de Vries (1991).

LEMMA 2.3. *Let* $M = \operatorname{conv}\{v^0, \ldots, v^n\}$ *be the vertex representation of an n-*

simplex in $\mathbb{R}^n$, *and let* ε_0 *be any number satisfying* $0 < \varepsilon_0 < 1/(n+1)$. *Let* $x^0 \in \operatorname{int} M$ *satisfy*

$$x^0 = \sum_{i=0}^{n} \lambda_i v^i , \quad \sum_{i=0}^{n} \lambda_i = 1 , \quad \lambda_i \geq \varepsilon_0 \quad (i = 0, \dots, n) ,$$

and consider

$$s^i = v^i + \mu_i (x^0 - v^i)$$

with $\mu_i \geq 1 + \varepsilon_0 \quad (i = 0, \dots . n)$.

Finally, let

$$H_i = \operatorname{aff}\{s^j : j = 0, \dots, n; j \neq i\} .$$

Then, for each intersection point

$$y^{ij} \in H_i \cap [v^j, v^i],\ y^{ij} = v^i + \lambda_{ij}(v^j - v^i)(i, j \in \{0, \dots, n\}, j \neq i)$$

it follows that

$$\lambda_{ij} < \frac{1 - \varepsilon_0}{1 - \varepsilon_0 + \varepsilon_0^2} .$$

THEOREM 2.2. *Let* $0 < \varepsilon_0 < 1/(n+1)$. *Then the successive cover introduced in Theorem 2.1, where an element of the form* $\bar{M}$ *never occurs in the decreasing sequence* $\{M_q\}$, *is exhaustive whenever the conditions on* x^0 *and* μ_i *of Lemma 2.3 are satisfied throughout the successive covering process.*

Proof. We show that under the conditions of Lemma 2.3, for the simplices M, $M_i(i \in \{0, \dots, n\})$ in Theorem 2.1 we have

$$\delta(M_i) < \frac{1 - \varepsilon_0}{1 - \varepsilon_0 + \varepsilon_0^2}\, \delta(M) , \tag{6}$$

which implies exhaustiveness since $(1 - \varepsilon_0)/(1 - \varepsilon_0 + \varepsilon_0^2) < 1$. Let $\alpha := (1 - \varepsilon_0)/(1 - \varepsilon_0 + \varepsilon_0^2)$, and consider y^{ij} and λ_{ij} as defined in Lemma 2.3. Fix an arbitrary i, and note that M_i has the vertices $\{v^i, y^{ij}(j \in \{0, \dots, n\} i \neq j)\}$. From $y^{ij} - v^i = \lambda_{ij}(v^j - v^i)$, $\lambda_{ij} < \alpha$ for all $j (j \neq i)$, we see, since $\delta(M)$ is the length of a longest edge of M, that for all $j \neq i$,

$$\| y^{ij} - v^i \| < \alpha \delta(M) .$$

It remains to show that $\| y^{ij} - y^{ik} \| < \alpha\delta(M)$ $(j \neq i, k \neq i, j \neq k)$. To see this, let $\bar{y}^{ij} = v^i + \alpha(v^j - v^i)$ and consider the triangle $\Delta = \operatorname{conv}[v^i, \bar{y}^{ij}, \bar{y}^{ik}] \subset M$ for arbitrary j, $k \in \{0, \dots, n\}$, $j \neq k$, $k \neq i$, $j \neq i$. Let $\delta(\Delta)$ denote the length of the longest edge of Δ. Since $\lambda_{ij} < \alpha$, we have $[y^{ij}, y^{ik}] \subset \Delta$, and

$$\| y^{ij} - y^{ik} \| < \delta(\Delta) . \tag{7}$$

When $\delta(\Delta) = \| v^i - \bar{y}^{ij} \|$ or $\delta(\Delta) = \| v^i - \bar{y}^{ik} \|$ it follows from the construction that $\delta(\Delta) = \alpha\delta(M)$, and hence the desired assertion follows from (7). When

$\delta(\Delta) = \| \bar{y}^{ij} - \bar{y}^{ik} \|$ we see from elementary geometry that

$$\frac{\| \bar{y}^{ij} - \bar{y}^{ik} \|}{\| v^j - v^k \|} = \frac{\| v^i - \bar{y}^{ij} \|}{\| v^i - v^j \|} = \alpha ,$$

and hence

$$\| y^{ij} - y^{ik} \| < \| \bar{y}^{ij} - \bar{y}^{ik} \| = \alpha \| v^j - v^k \| \leq \alpha \delta(M) . \qquad \square$$

3. The Algorithm

We recall some known definitions and additional results which will aid in the presentation and understanding of the algorithm. Numerical experiments clearly indicate that it is worthwhile to invest in finding a tight initial simplex $M_0 \supset D$ whose facets support D. One way to find such a simplex is to solve the following $n + 1$ convex minimization problems (with linear objective function):

$$\alpha_j := \min\{x_j : x \in D\} \quad (j = 1, \ldots, n)$$

and

$$\alpha := \max\left\{ \sum_{j=1}^{n} x_j : x \in D \right\}$$

to obtain

$$M_0 = \left\{ x \in \mathbb{R}^n : \alpha_j - x_j \leq 0 \ (j = 1, \ldots, n) . \ \sum_{j=1}^{n} (x_j - \alpha) \leq 0 \right\} .$$

The facets $\{x \in \mathbb{R}^n : x_j = \alpha_j\}$ $(j = 1, \ldots, n)$, $\{x \in \mathbb{R}^n : \Sigma_{j=1}^n x_j = \alpha\}$ of M support D, and the vertices $v^0, \ldots, v^n$ of M_0 have the form

$$v^0 = (\alpha_1, \ldots, \alpha_n)^T , \quad v^j$$
$$= (\alpha_1, \ldots, \alpha_{j-1}, \bar{\alpha}_j, \alpha_{j+1}, \ldots, \alpha_n) \quad (j = 1, \ldots, n)$$

where $\bar{\alpha}_j = \alpha - \Sigma_{i \neq j} \alpha_i$.

In addition to the successive cover discussed in Section 3 we will also use the classical bisection of simplices.

DEFINITION 3.1. Let $M = \text{conv}\{v^0, \ldots, v^n\}$ be the vertex representation of an n-simplex M, and let $w = \frac{1}{2}(v^r + v^s)$ be the midpoint of a longest edge $[v^r, v^s]$ of M. The partition $\{M_1, M_2\}$ of M into two simplices $M_1 = \text{conv}\{v^0, \ldots, v^{r-1}, w, v^{r+1}, \ldots, v^n\}$ and $M_2 = \text{conv}\{v^0, \ldots, v^{s-1}, w, v^{s+1}, \ldots, v^n\}$ is called *bisection* of M.

Bisection of simplices was introduced in Horst (1976) and since then used in many simplicial branch and bound algorithms (cf., e.g., Horst and Tuy, 1990 and references therein). Different proofs of the following lemma are given in Thoai and Tuy (1980) and in Tuy, Khatchaturov, and Utkin (1987). Comprehensive discussions of bisection and more general simplicial partitions are given in Horst and Tuy (1990).

LEMMA 3.1. *Successive bisection of simplices is exhaustive, i.e., any infinite decreasing (nested) sequence* $\{M_q\}$ *of simplices generated by successive bisection converges to a singleton:*

$$\bigcap_q M_q = \lim_{q\to\infty} M_q = \{\bar{x}\} .$$

Next, we recall the definition of a convex envelope.

DEFINITION 3.2. The convex envelope of a function taken over a nonempty subset of S its domain is that function φ_s such that

(i) φ_s is convex defined over the convex hull of S;

(ii) if h is a convex function defined over the convex hull of S that satisfies $h(x) \leqslant f(x)$ for all $x \in S$, then $h(x) \leqslant \varphi_S(x)$ for any x in the convex hull of S.

A survey of various interesting properties of convex envelopes is given in Horst and Tuy (1990).

The convex envelope φ_M of f taken over a simplex M, is the unique affine function that coincides with f at the vertices of M (cf. Section 1). Moreover, it follows immediately from Definition 3.2 that $S_1 \subset S_2$ implies $\varphi_{S_1} \geqslant \varphi_{S_2}$ on S_1 (monotonicity of φ_S, cf. Horst and Tuy, 1990; Benson and Horst, 1991. In each iteration of the algorithm, we will have a simplicial cover $\mathcal{M}$ of a collection of simplices which are still candidates to contain an optimal solution of (P). We will also have an outer polytope-approximation $P \supset D$ of the feasible set D. For each simplex $M \in \mathcal{M}$, a lower bound $\beta(M)$ for $f(x)$ over $M \cap D$ will be determined by solving the linear programming problem

$$\beta(M) := \min\{\varphi_M(x) : x \in M \cap P\} . \tag{8}$$

Another useful concept in concave minimization is the so-called γ-*extension* which was introduced in Tuy (1964), and since then successfully used only in various conical branch and bound algorithms (e.g. Thoai and Tuy, 1980; Horst and Thoai, 1989; Horst and Tuy, 1990; Horst, Thoai, and Benson, 1991).

The following definition generalizes the classical γ-extension so that it can be used in the simplicial cover technique to eliminate simplices of type $\bar{M}$ in the cover defined in Theorem 2.1.

DEFINITION 3.3. Let $f: \mathbb{R}^n \to \mathbb{R}$ be concave; $v, x \in \mathbb{R}^n$, $x \neq v$, $\gamma \in \mathbb{R}$ satisfying $\gamma \leqslant f(x)$, and $\theta_1 > 1$ be an arbitrary large number. Define $\theta_0 = \max\{\mu \geqslant 1 : f(v + \mu(x - v)) \geqslant \gamma\}$, and set

$$\theta = \begin{cases} \theta_0 & \text{if } \theta_0 \text{ exists} \\ \theta_1 & \text{otherwise} \end{cases}$$

Then the point $s = v + \theta(x - v)$ is called *γ-extension* of x with respect to v.

The geometric meaning of the notion of a γ-extension is as follows. Assume first that the level set $L_\gamma := \{y: f(y) \geq \gamma\}$ is bounded. Then θ_0 exists, and the γ-extension of x with respect to v is the farthest intersection point with respect to v of the ray $\rho(x, v)$ from v through x with the boundary $\{y: f(y) = \gamma\}$ of L_γ. Note that $x \in L_\gamma$, but not necessarily $v \in L_\gamma$ so that $\rho(x, v)$ might intersect $\{y: f(y) = \gamma\}$ twice. The number θ_1 replaces θ_0 in the case of unbounded L_γ when θ_0 does not exist. Since L_γ is convex computing θ_0 amounts to solving a univariate convex minimization problem (line-search) which can be done in various ways (cf. Horst and Thoai, 1989; Horst, Thoai, and Benson, 1991).

In the algorithm below, the number γ will always be the objective function value at the best feasible point obtained so far (best feasible upper bound of m). Next, we show that, in the cover introduced in Section 2, the "inner" simplex $\bar{M}$ can always be deleted when γ-extensions are used.

LEMMA 3.2. *Let $M = \mathrm{conv}\{v^0, \ldots, v^n\}$ be the vertex representation of an n-simplex in $\mathbb{R}^n$, and let $x^0 \in \mathrm{int}\, M$ satisfy $f(x^0) > \gamma \geq m$. Consider the simplex $\bar{M} = \mathrm{conv}\{s^0, \ldots, s^n\}$, where s^i is the γ-extension of x^0 with respect to v^i ($i = 0, \ldots, n$). Then*

$$f(x) \geq m \quad \forall x \in \bar{M}\,.$$

Proof. From the definition of a γ-extension we see that $f(s^i) \geq m$ ($i = 0, \ldots, n$). The assertion follows, since the minimum value of f over $\bar{M}$ is attained at a vertex of $\bar{M}$. □

The following algorithm, one of several versions of which is given below in some detail, constructs two sequences $\{x^k\}$ and $\{r^k\}$ of points and two associated sequences $\{\gamma_k\}$ and $\{\beta_k\}$ of real numbers where

(i) $x^k \in D$ (the incumbent best feasible solution), and $\gamma_k = f(x^k)$;
(ii) $\beta_k \leq \min\{f(x): x \in D\}$, and r^k satisfying $\varphi_M(r^k) = \beta_k$ for some simplex M of the current simplicial cover.

Iteration 0.

0.1. Choose $0 < \varepsilon_0 < 1/(1+n)$, $\varepsilon_1 \geq 0$. Find a point p such that $g(p) < 0$. Find an n-simplex $M_0 = \mathrm{conv}\{v^0, \ldots, v^n\}$, $D \subset M_0$, whose facets support D.

0.2. Determine

$$z^i = p + \lambda_i(p - v^i), \text{ where } \lambda_i = \max\{\theta: g(p + \theta(p - v^i)) \leq 0\}$$
$$(i = 0, \ldots, n\}\,,$$

$$z^{n+1+i} = p - \lambda_{n+1+i}(p - v^i), \text{ where } \lambda_{n+1+i} = \max\{\theta\colon g(p - \theta(p - v^i)) \leq 0\} \quad (i = 0, \ldots, n\} ,$$

and set

$$\gamma_0 = \min\{f(p), f(z^i)\colon i = 0, \ldots, 2(n+1)\} , \quad y^0 \in D\colon f(y^0) = \gamma_0 .$$

0.3. For $i = 0, \ldots, 2(n+1)$, find

$$\xi^i \in \partial g(z^i) , \quad H_i := \xi^{i^T}(x - z^i) \leq 0\} .$$

Set

$$P := M_0 \cap \left(\bigcap_{i=0}^{2(n+1)} H_i \right) .$$

0.4. For $i = 0, \ldots, n$, determine the γ_0-extensions s^i of z^i with respect to p.

0.5. Construct the simplicial cover $\mathcal{M} = \{M_i\colon i \in I_0, I_0 \subset \{1, \ldots, n+2\}\}$ of M_0 defined in Theorem 2.1 with respect to s^i $(i = 0, \ldots, n)$ (where $x^0 = p$). If $|I_0| = 1$, then stop: y^0 is an optimal solution of (P) (cf. Lemma 3.2)

0.6. For each $M \in \mathcal{M}$ find the convex envelope φ_M of f over M, and solve the linear program

$$\min\{\varphi_M(x)\colon x \in M \cap P\}$$

obtaining the optimal value $\beta(M) = \varphi_M(r(M))^1$ with an optimal solution $r(M)$ when $\beta(M) < \infty$. Set $\beta_0 = \min\{\beta(M)\colon M \in \mathcal{M}\}$. Choose $\tilde{M}$ satisfying $\beta_0 = \beta(\tilde{M})$, and set $r^0 = r(\tilde{M})$.

Iteration $k(k \geq 1)$.

k.1. Set $\mathcal{R} := \{M \in \mathcal{M}\colon \beta(M) < \gamma_{k-1} - \varepsilon_1\}$.

k.2. If $\mathcal{R} = \emptyset$, then stop: y^{k-1} is an ε_1-optimal solution, and $\gamma_{k-1} - \beta_{k-1} < \varepsilon_1$.

k.3. (Refining $\tilde{M}$). Let $\tilde{M} = \text{conv}\{w^0, \ldots, w^n\}$. If $r^{k-1} \in D$, then set $\gamma_{k-1} = \min\{\gamma_{k-1}, f(r^{k-1})\}$.

k.3.1. If $r^{k-1} \not\in D$ and $f(r^{k-1}) \leq \gamma_{k-1}$, then bisect $\tilde{M}$ into two simplices $\tilde{M}_1$ and $\tilde{M}_2$. Go to *k.4*.

k.3.2. If in the representation

$$r^{k-1} = \sum_{j=0}^{n} \lambda_j w^j , \quad \sum_{j=0}^{n} \lambda_j = 1 ,$$

there is a $j \in \{0, \ldots, n\}$ such that $\lambda_j < \varepsilon_0$, then bisect $\tilde{M}$ into two simplices $\tilde{M}_1$, $\tilde{M}_2$. Go to *k.4*.

k.3.3. Find the γ_{k-1}-extensions $s^{i,k-1}$ of r^{k-1} with respect to w^i, say

$$s^{i,k-1} = w^i + \mu_i(r^k - w^i) \quad (i = 0, \ldots, n) .$$

If $\mu_i > 1 + \varepsilon_0$ for all $i \in \{0, \ldots, n\}$, then construct the simplicial cover of $\tilde{M}$ defined in Theorem 2.1 with respect to $s^{i,k-1}$ $(i = 0, \ldots, n)$. Otherwise, bisect $\tilde{M}$ into two simplices $\tilde{M}_1$, $\tilde{M}_2$.

k.4. (Improvement of the polytope approximation P). If $r^{k-1} \notin D$, determine

$$\tilde{z}^k = p + \lambda_k(r^{k-1} - p)$$

where

$$\lambda_k = \max\{\theta : g(p + \theta(r^{k-1} - p)) \leq 0\} .$$

Find

$$\tilde{\xi}^k \in \partial g(\tilde{z}^k)$$

and set

$$P = P \cap \{x \in \mathbb{R}^n : (\tilde{\xi}^k)^T(x - \tilde{z}^k) \leq 0\} .$$

k.5. (new lower bounds). Let $\tilde{\mathcal{M}}$ denote the cover of $\tilde{M}$ obtained in Step *k.3*. For each $M \in \tilde{\mathcal{M}}$ find the convex envelope φ_M of f over M, and solve the linear program

$$\min\{\varphi_M(x) : x \in M \cap P\}$$

obtaining the optimal value $\beta(M) = \varphi_M(r(M))$ with an optimal solution $r(M)$ when $\beta(M) < \infty$.

k.6. Set $\mathcal{R} = \mathcal{R} \backslash \{\tilde{M}\} \cup \tilde{\mathcal{M}}$, $\gamma_k := \min\{\gamma_{k-1}, f(z) : z \in S^k\}$, $y^k : f(y^k) = \gamma_k$, where $S^k \subset D$ is the finite set of feasible points obtained while carrying out the preceding steps in iteration k.

Set $\beta_k := \min\{\beta(M) : M \in \mathcal{R}\}$. Choose $\tilde{M}$ satisfying $\beta_k = \beta(\tilde{M})$ and set $r^k = r(\tilde{M})$. Go to iteration $k + 1$.

REMARKS. (1) Steps 0.1, 0.2 and 0.3 take into account the numerical experience that it is worthwhile to invest some computational effort in a good initial outer approximation P and good initial upper and lower bound γ_0 and β_0, respectively.

(2) If in step *k.3* we have $r^{k-1} \in \partial\tilde{M}$, then it is often advantageous to replace in this step r^{k-1} by the barycenter of $\tilde{M}$.

(3) The set S^k in *k.6.* contains at least $\tilde{z}^k$ and all new $r(M)$ satisfying $r(M) \in D$.

(4) In *k.5*, additional cuts can be introduced similarly to steps 0.2 and 0.3 in order to improve the outer approximation P of D, where, however, an appropriate balance with the numerical effort caused by the size of the linear programs in step *k.5.* For constraint-dropping strategies, see Horst and Tuy (1990) and references therein.

The convergence of the algorithm can now be shown.

THEOREM 3.1. *If the above algorithm with* $\varepsilon_1 = 0$ *does not terminate after a finite number of iterations, then we have*

$$\gamma := \lim_{k\to\infty} \gamma_k = \lim_{k\to\infty} f(y^k) = \min\{f(x) : x \in D\} = \lim_{k\to\infty} \beta_k := \beta ,$$

and every limit point of the sequence $\{y^k\}$ *and also every limit point of the sequence* $\{r^k\}$ *is an optimal solution of problem (P).*

Proof. As above, denote $m = \min\{f(x): x \in D\}$. One way to prove Theorem 3.1 is to demonstrate that the convergence conditions of the general branch and bound theory developed in Horst (1986), Horst and Tuy (1987), Tuy and Horst (1988) are fulfilled. Since outer approximation is involved we would also have to refer to general outer approximation theories, as, e.g., presented in Eaves and Zangwill (1971) and more recently by Horst, Thoai and Tuy (1987 and 1989) (for a comprehensive discussion of both theories, see Horst and Tuy, 1990). It is convenient, however, to give a brief direct proof here which only assumes the reader to be familiar with the convergence properties of the supporting hyperplane-outer approximation method used in Step *k.4*.

Note that the sequence $\{\gamma_k\}$ of upper bounds is nonincreasing and bounded from below by m. Likewise, by the monotonicity property of convex envelopes stated above, the sequence $\{\beta_k\}$ of lower bounds is nondecreasing and bounded from above by m. It follows that both limits γ and β exist, and

$$\gamma \geqslant m \geqslant \beta \,. \tag{9}$$

In the first part of the proof we show that $\beta = m$, and every limit point $\bar{r}$ of $\{r^k\}$ is an optimal solution of (P). The second part considers γ and the sequence $\{y^k\}$.

Part I. We first show in Part I.1 that $f(\bar{r}) \leqslant m$. In part I.2 we prove that $\bar{r} \in D$, and hence $f(\bar{r}) \geqslant m$.

Part I.1. Denote by $\{r^q\}$ a subsequence of $\{r^k\}$ satisfying $r^q \underset{q\to\infty}{\longrightarrow} \bar{r}$. Then we conclude, using a standard argument on the finiteness of the number of partition elements in each iteration (see e.g., Horst, 1976, 1986, Horst and Tuy, 1987 and 1990), that there exists an infinite decreasing subsequence $\{\tilde{M}^{q'}\} \subseteq \{\tilde{M}^q\}$ such that either $\tilde{M}^{q'+1}$ is generated from $\tilde{M}q'$ by bisection or else $\tilde{M}^{q'+1}$ is an element of the simplicial cover of $\tilde{M}^{q'}$ as defined in Theorem 2.1 (cf. Step *k.3*). From Theorem 2.2 and Lemma 2.1 we know that there is a point $\tilde{r}$ satisfying $\tilde{M}^{q'} \underset{q'\to\infty}{\longrightarrow} \{\tilde{r}\}$. But, by definition of $r^{q'}$ and $\tilde{M}^{q'}$, respectively, we have $r^{q'} \in \tilde{M}^{q'}$. It follows that $r^{q'} \to \tilde{r}$, and hence $\bar{r} = \tilde{r}$.

Next, we recall that the convex envelope $\varphi_{\tilde{M}^{q'}}$ of f over $\tilde{M}^{q'}$ is the affine function that coincides with f at the vertices of $\tilde{M}^{q'}$. Since both f and $\varphi_{\tilde{M}^{q'}}$ attain their global minima over $\tilde{M}^{q'}$ at a vertex of $\tilde{M}^{q'}$, we have

$$\begin{aligned}\min\{f(x): x \in \tilde{M}^{q'}\} &= \min\{\varphi_{\tilde{M}^{q'}}(x): x \in \tilde{M}^{q'}\} \\ &\leqslant \min\{\varphi_{\tilde{M}^{q'}}(x): x \in P^{q'} \cap \tilde{M}^{q'}\} \\ &= \varphi_{\tilde{M}^{q'}}(r^{q'}) = \beta_{q'} \,.\end{aligned} \tag{10}$$

Letting $q' \to \infty$, we see, by continuity of f, that $\min\{f(x): x \in \tilde{M}^{q'}\} \underset{q'\to\infty}{\longrightarrow} f(\bar{r})$, and (9), (10) yield

$$f(\bar{r}) \leqslant \beta \leqslant m \,. \tag{11}$$

Part I.2. By (11) we see that Part I of the proof is established if $\bar{r} \in D$. Let as above $\{r^q\}$ be a subsequence satisfying $r^q \xrightarrow[q\to\infty]{} \bar{r}$. If $\{r^q\}$ contains a subsequence of points all of which lie in D, then by closedness of D we must have $\bar{r} \in D$. Therefore assume without loss of generality that $r^q \subset \mathbb{R}^n \backslash D$. But in this case, we know from the theory of the outer approximation methods of Step *k.4* (supporting hyperplane-method) that

$$\bar{r} \in D, \text{ and also } \tilde{z}^q \xrightarrow[q\to\infty]{} \bar{r}\,, \tag{12}$$

where $\tilde{z}^q$ is defined in Step *k.4*. For a proof, see, e.g., Benson and Horst (1991), Horst and Tuy (1990) and references there.

Part II. Let $\bar{y}$ denote a limit point of $\{y^k\}$, and let $\{y^r\}$ be a subsequence of $\{y^k\}$ satisfying $y^r \xrightarrow[q\to\infty]{} \bar{y}$. Consider the corresponding sequence $\{\tilde{z}^r\} \subset \partial D$. Since ∂D is compact there exists a convergent subsequence $\{\tilde{z}^q\}$ of $\{\tilde{z}^r\}$ that, by (12) satisfies $\tilde{z}^q \xrightarrow[q\to\infty]{} \bar{r}$. Since $\tilde{z}^q \in S^q \subset D$ (cf. step *k.6*) we must have $m \leq \gamma_q = f(y^q) \leq f(\tilde{z}^q)$. But in Part I we have seen that $f(\tilde{z}^q) \xrightarrow[q\to\infty]{} f(\bar{r}) = m$. It follows, by continuity of f and by (9), that $\gamma = f(\bar{y}) = m$.

4. Numerical Aspects and Illustrative Example

In this section some brief comments are given on the computational subprocedures called for by the algorithm. An illustrative example is presented, and first results of numerical experiments are reported.

Each of the computational procedures called for by the algorithm has been implemented and discussed previously. For the computation of the initial simplex M_0 and of the convex envelopes we refer to Section 3 of this article and the references given there. The relevant information and the references for the remaining procedures are given in Table I.

In the implementation of the lower bounding LP-procedure for solving

$$\min\{\varphi_M(x) : x \in M \cap P\}\,, \tag{13}$$

problem (13) is transformed by the one-to-one linear mapping B which maps a simplex $M = \text{conv}\{w^0, \ldots, w^n\}$ onto the standard simplex $S := \{x \in \mathbb{R}^n : x_j \geq 0$ $(j = 1, \ldots, n),\ \Sigma_{j=1}^n x_j \leq 1\}$. It is easy to see that B is represented by the $n \times n$ matrix B with columns $(w^i - w^0)$ $(i = 1, \ldots, n)$. The numerical advantage of such a transformation is similar to that of corresponding transformation of cones discussed in Horst and Thoai (1989).

To illustrate the new algorithm for solving (P), we apply it to the small example which was considered in Benson and Horst (1991). A comparison of the bounds γ_k and β_k, respectively, of the new algorithm with the corresponding bounds γ_k^{BH} and β_k^{BH} of the Benson–Horst algorithm shows the anticipated improvement achieved by the new cover technique. Consider the problem

Table I.

Subproblem	Procedure	Reference
γ-extension	line-search	Horst and Thoai (1989) Horst and Tuy (1990) Horst, Thoai, and Benson (1991)
intersection of a ray or line-segment with the boundary ∂D of D	univariate convex minimization problem	Horst and Thoai (1989) Horst and Tuy (1990) Benson and Horst (1991) Horst, Thoai, and Benson (1991)
subgradient of g at z	convex combination of the gradients $\nabla g_i(z)$ satisfying $g_i(z) = g(z)$	Benson and Horst (1991)
lower bounds $\min\{\varphi_M(x): x \in M \cap P\}$	simplex and dual-simplex	Benson and Horst (1991)

$$\begin{aligned} \text{minimize} \quad & f(x_1, x_2) = -129x_1^2 + 242x_1x_2 - 129x_2^2 + 1258x_1 + 1242x_2 \\ \text{s.t.} \quad & -x_1 - x_2 - 2 \leqslant 0 \\ & -4x_1 + x_2^2 - 8 \leqslant 0 \\ & 16x_1^2 - 32x_1 + 25x_2^2 - 384 \leqslant 0 . \end{aligned}$$

The exact optimal solution is given by $x^* = (-1, 2)$ with $m = f(x^*) = -4871$. We choose the interior point $p = (0, 0)$ and stop when we are guaranteed that γ_k is within 3% of m (cf. Benson and Horst, 1991). The initial simplex

$$\begin{aligned} M_0 &= \operatorname{conv}\{(11.33; -3.93), (-2.00; 9.40), (-2.00; -3.93)\} \\ &= \{x \in \mathbb{R}^2, x_1 \geqslant -2, x_2 \geqslant -3.93, x_1 + x_2 \leqslant 7.43\} , \end{aligned}$$

and the initial polytope

$$\begin{aligned} P = \{x\colon & 141.27x_1 - 93.89x_2 \leqslant 941.27; -4.00x_1 + 1.31x_2 \leqslant 8.43; \\ & -4.00x_1 + 4.87x_2 \leqslant 13.93; -x_1 - x_2 \leqslant 2.00; 31.83x_1 + 196.00x_2 \\ & \leqslant 831.83\} \end{aligned}$$

yield the following first simplicial cover, corresponding convex envelopes and lower bounds:

$$\begin{aligned} M_{0,1} &= \operatorname{conv}\{(11.33; -3.93); (2.88; 4.52); (2.52; -3.93)\} , \\ \varphi_{M_{0,1}}(x) &= -1490.52x_1 - 573.82x_2 + 4321.99, \ \beta(M_{0,1}) = -6234.95 \\ M_{0,2} &= \operatorname{conv}\{(.228; 5.13); (-2.00; 9.40); (-2.00; 0.94)\} , \\ \varphi_{M_{0,2}}(x) &= 3002.66x_1 - 3060.06x_2 + 4111.85, \ \beta(M_{0,2}) = -5423.44 \end{aligned}$$

$$M_{0,3} = \text{conv}\{(-0.35;\ -3.93);\ (-2.00;\ 0.29);\ (-2.00;\ -3.93)\}\ ,$$

$$\varphi_{M_{0,3}}(x) = 609.77x_1 - 1255.89x_2 - 1957.42,\ \beta(M_{0,3}) = -3536.06,$$

(cf. Fig. 1).

At the end of the initial iteration 0 we have $y^0 = (-0.52;\ 2.43)$, $\gamma_0 = f(y^0) = -4780.10$, $r^0 = (6.96;\ 0.44)$, and $\beta_0 = \beta(M_{0,1}) = -6234.95$; whereas the Benson–Horst method obtained $\beta_0^{BH} = -19032$, $\gamma_0^{BH} = 0$.

In iteration 1, the simplex $M_{0,3}$ is deleted, and the simplex $M_{0,1}$ is bisected to yield

$$M_{1,1} = \text{conv}\{(7.11;\ 0.29);\ (2.88;\ 4.52);\ (2.52;\ -3.93)\},$$

$$\varphi_{M_{1,1}}(x) = 545.54x_1 - 660.20x_2 - 1129.23\ ,$$

and

$$M_{1,2} = \text{conv}\{(11.33;\ -3.93);\ (7.11;\ 0.29);\ (2.52,\ -3.93)\}\ ,$$

$$\varphi_{M_{1,2}}(x) = 1480.52x_1 + 1538.62x_2 + 12624.23\ .$$

Since $r^0 \not\in D$, we set $P := P \cap \{x\colon 159.27x_1 + 19.09x_2 \leq 959.27\}$, and obtain $\beta(M_{1,1}) = -2069.61$, $\beta(M_{1,2}) = 579.92$.

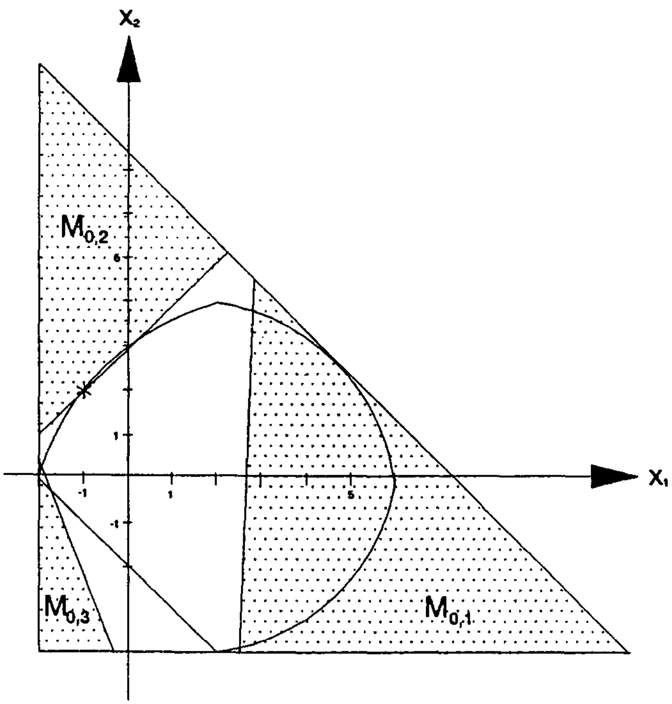

Fig. 1. Initial simplex cover.

At the end of iteration 1, we have $y^1 = y^0$, $\gamma_1 = \gamma_0 = -4780.10$, $r^1 = (-1.60; 1.55)$ and $\beta_1 = \beta(M_{0,2}) = -5423.44$, whereas then Benson–Horst approach yields $\beta_1^{BH} = -16120$, $\gamma_1^{BH} = -3032$.

In iteration 2, the simplices $M_{1,1}$ and $M_{1,2}$ are deleted, the simplex $M_{0,2}$ is bisected to yield

$$M_{2,1} = \text{conv}\{(2.28; 5.13); (-2.00; 5.17); (-2.00; 0.94)\}\ ,$$

$$\varphi_{M_{1,1}}(x) = 2468.29x_1 - 2514.11x_2 + 2530.67$$

and

$$M_{2,2} = \text{conv}\{(2.28; 5.13); (-2.00; 9.40); (-2.00; 5.17)\}\ ,$$

$$\varphi_{M_{2,2}}(x) = 2456.72x_1 - 3605.99x_2 + 8153.34\ .$$

Since $r^1 \not\in D$, we set now $P := P \cap \{x: -400x_1 + 2.87x_2 \leq 10.00\}$, and obtain $\beta(M_{2,1}) = -5114.58$ and $\beta(M_{2,2}) = -4864.56$. At the end of iteration 2 we have the point $y^2 = (-1.10;\ 1.89)$ and the bounds $\beta_2 = -5423.44$, $\gamma_2 = f(y^2) = -4865.56$. The bounds after iteration 2 of the Benson–Horst approach were $\beta_2^{BH} = -8532$, $\gamma_2^{BH} = -4871$.

The algorithm continues with bisections for only two more full iterations to obtain the required 3%-accuracy. At the end of iteration 4 we have $\beta_4 = -4963.04$, $r^4 = (-0.85;\ 2.16)$, $y^4 = (0.97;\ 2.02)$, and $\gamma^4 = f(y^4) = -4870.80$, whereas the Horst–Benson method stopped after 7 iterations with the bounds $\gamma_7^{BH} = -4871$ and $\beta_7^{BH} = -5016$.

After step 5.1 only the simplex $M = \text{conv}\{(0.14; 3.03),\ (-2.00, 3.05),\ (-2.00; 0.94)\}$ remains in the list of undeleted simplices. All of the other previously generated simplices were deleted (fathomed) in some step *k.1* ($k \in \{1, \ldots, 5\}$).

A similar superiority over the pure Benson–Horst approach was observed at 25 additional examples taken from the test examples described in Horst, Thoai, and Benson (1991): it is clearly worthwhile to invest the computational effort required in iteration 0 to obtain good initial bounds. We also observed in many examples that, after the initial iteration 0 only bisections were carried out in steps *k.3* ($k = 1, 2, \ldots$). Moreover, when successive simplicial covering was performed in exceptional cases, then the new algorithm was not guaranteed to be more efficient than the variant which, after one or two iterations, switched to a pure bisection.

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Note

[1]We adopt the usual convention that $\beta(M) = \infty$ when $M \cap P = \emptyset$. In this case, the simplex M is of course deleted from further consideration (cf. Step *k.1*.)

References

Al-Khayyal, F. A. and Falk, J. E. (1983), Jointly Constrained Biconvex Programming, *Mathematics of Operations Research* **8**, 273–286.

Benson, H. P. (1990), Separable Concave Minimization via Partial Outer Approximation and Branch and Bound, *Operations Research Letters* **9**, 389–394.

Benson, H. P. and Erenguc, S. (1988), Using Convex Envelopes to Solve the Interactive Fixed Charge Linear Programming Problem, *Journal of Optimization Theory and Applications* **59**, 223–246.

Benson, H. P. and Horst, R. (1991), A Branch and Bound-Outer Approximation Algorithm for Concave Minimization over a Convex Set, *J. Computers and Mathematics with Applications* **21**, 67–76.

Chen, P. C., Hansen, P., and Jaumard, B. (1991), On-Line and Off-Line Vertex Enumeration by Adjacency Lists, to appear in *Operations Research Letters*.

Eaves, B. C. and Zangwill, W. I. (1971), Generalized Cutting Plane Algorithms, *SIAM Journal on Control* **9**, 529–542.

Falk, J. E. and Soland, R. M. (1969), An Algorithm for Separable Nonconvex Programming Problems, *Management Science* **15**, 550–569.

Hoffman, K. L. (1981), A Method for Globally Minimizing Concave Functions over Convex Sets, *Mathematical Programming* **20**, 22–32.

Horst, R. (1976), An Algorithm for Nonconvex Programming Problems, *Mathematical Programming* **10**, 312–321.

Horst, R. (1980), A Note on the Convergence of an Algorithm for Noncovex Programming Problems, *Mathematical Programming* **19**, 237–238.

Horst, R. (1986), A General Class of Branch and Bound Methods in Global Optimization with Some New Approaches for Concave Minimization, *Journal of Optimization Theory and Applications* **51**, 271–291.

Horst, R. and Thoai, N. V., (1989), Modification, Implementation and Comparison of Three Algorithms for Globally Solving Linearly Constrained Concave Minimization Problems, *Computing* **42**, 271–289.

Horst, R., Thoai, N. V., and Benson, H. P. (1991), Concave Minimization via Conical Partitions and Polyhedral Outer Approximation, to appear in *Mathematical Programming*.

Horst, R., Thoai, N. V., and de Vries, J. (1988), On Finding New Vertices and Redundant Constraints in Cutting Plane Algorithms for Global Optimization, *Operations Research Letters* **7**, 85–90.

Horst, R., Thoai, N. V., and de Vries, J. (1991), On Geometry and Convergence of Simplicial Covers, submitted for publication.

Horst, R., Thoai, N. V., and Tuy, H. (1987), Outer Approximation by Polyhedral Convex Sets. *Operations Research Spektrum* **9**, 153–159.

Horst, R., Thoai, N. V., and Tuy, H. (1989), On an Outer Approximation Concept in Global Optimization, *Optimization* **20**, 255–264.

Horst, R. and Tuy, H. (1987), On the Convergence of Global Methods in Multiextremal Optimization, *Journal of Optimization Theory and Applications* **54**, 253–271.

Horst, R. and Tuy, H. (1990), *Global Optimization* (*Deterministic Approaches*), Springer, Berlin.

Pardalos, P. M. and Rosen, J. B. (1987), Constrained Global Optimization: Algorithms and Applications, *Lecture Notes in Computer Science* **268**, Springer-Verlag, Berlin.

Thieu, T. V., Tam, B. T., and Ban, T. V. (1983), An Outer Approximation Method for Globally Minimizing a Concave Function over a Compact Convex Set, *Acta Mathematica Vietnamica* **8**, 21–40.

Thoai, N. V. and de Vries, J. (1988), Numerical Experiments on Concave Minimization Problems, Proceedings XIII. Symposium on Operations Research, *Methods of Operations Research* **60**, 363–365.

Tuy, H. (1991), Effect of the Subdivision Strategy on Convergence and Efficiency of Some Global Optimization Algorithms, *J. of Global Optimization* **1**, 23–36.

Tuy, H. and Horst, R. (1988), Convergence and Restart in Branch and Bound Algorithms for Global Optimization, Application to Concave Minimization and DC-Optimization Problems, *Mathematical Programming* **41**, 161–183.

Tuy, H., Thieu, T. V., and Thai, N. Q. (1985), A Conical Algorithm for Globally Minimizing a Concave Function Over a Closed Convex Set, *Mathematics of Operations Research* **10**, 498–515.

Veinott, A. F. (1967), Minimum Concave Cost Solution of Leontiev Substitution Models of Multi-Facility Inventory Systems, *Operations Research* **14**, 486–507.